Solution for HW3

4-10-2016

§20) 4)
$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \lim_{z\to z_0}\frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}} = \frac{\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}}{\lim_{z\to z_0}\frac{g(z)\cdot g(z_0)}{z-z_0}} = \frac{f'(z_0)}{g'(z_0)}$$

8) a) Suppose the derivative exists at some point $z_0 = x_0 + iy_0$.
Approaching z_0 horizontally, we have $z = (x_0+\Delta x) + iy_0$ and
$$\frac{Re(z)-Re(z_0)}{z-z_0} = \frac{\Delta x}{\Delta x} = 1$$
Approaching z_0 vertically, we have $z = x_0 + i(y_0+\Delta y)$ and
$$\frac{Re(z)-Re(z_0)}{z-z_0} = \frac{0-0}{i\Delta y} = 0 \neq 1$$, contradiction.
Hence $f'(z)$ does not exist at any point z.

b) Suppose the derivative exists at some point $z_0 = x_0 + iy_0$
Approaching z_0 horizontally, we have $z = (x_0+\Delta x) + iy_0$ and
$$\frac{Im(z)-Im(z_0)}{z-z_0} = \frac{0-0}{\Delta x} = 0$$
Approaching z_0 vertically, we have $z = x_0 + i(y_0+\Delta y)$ and
$$\frac{Im(z)-Im(z_0)}{z-z_0} = \frac{\Delta y}{i\Delta y} = \frac{1}{i} \neq 0$$, contradiction.
Hence $f'(z)$ does not exist at any point z.

§23) 1) a) $f(z) = \bar{z} = x - iy$.
So we have $u(x,y) = x$ & $v(x,y) = -y$.
Since $u_x = 1 \neq -1 = v_y$ for any x, y, $f'(z)$ does not exist for any z.

c) $f(z) = 2x + ixy^2$.
So we have $u(x,y) = 2x$ & $v(x,y) = xy^2$.
The Cauchy-Riemann equation holds
$$\Leftrightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Leftrightarrow \begin{cases} 2 = 2xy \\ 0 = -y^2 \end{cases}$$, which is impossible.

Hence $f'(z)$ does not exist for any z.

2) c) $f(z) = z^3 = (x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$.

Hence $u(x,y) = x^3 - 3xy^2$ and $v(x,y) = 3x^2y - y^3$.

Note that u & v exist and are continuous everywhere.

Furthermore, $\begin{cases} u_x = 3x^2 - 3y^2, & v_x = 6xy \\ u_y = -6xy & v_y = 3x^2 - 3y^2 \end{cases}$

$\Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$\Rightarrow$ $f'(z)$ exists everywhere.

$f'(z) = u_x + iv_x = 3x^2 - 3y^2 + i(6xy)$.

Note that $\begin{cases} (3x^2 - 3y^2)_x = 6x = (6xy)_y \\ (3x^2 - 3y^2)_y = -6y = -(6xy)_x \end{cases}$

Hence $f''(z)$ exists everywhere with

$f''(z) = (3x^2 - 3y^2)_x + i(6xy)_x = 6x + 6iy$

d) $f(z) = \cos x \cosh y - i \sin x \sinh y$

Hence $u(x,y) = \cos x \cosh y$, $v(x,y) = \sin x \sinh y$.

Note that u & v exist and are continuous everywhere.

Furthermore, $\begin{cases} u_x = -\sin x \cosh y = v_y \\ u_y = \cos x \sinh y = -v_x \end{cases}$

$\Rightarrow f'(z)$ exists everywhere with

$f'(z) = u_x + iv_x = -\sin x \cosh y - i \cos x \sinh y$

Similarly, note that $\begin{cases} (-\sin x \cosh y)_x = -\cos x \cosh y = (-\cos x \sinh y)_y \\ (-\sin x \cosh y)_y = -\sin x \sinh y = -(-\cos x \sinh y)_x \end{cases}$

Hence $f''(z)$ exists everywhere with

$f''(z) = -\cos x \cosh y + i \sin x \sinh y$.

4) a) $f(z) = z^{-4} = r^{-4} e^{-4i\theta} = r^{-4} \cos 4\theta - r^{-4} i \sin 4\theta$.

So $u(r,\theta) = r^{-4} \cos 4\theta$, $v(r,\theta) = -r^{-4} \sin 4\theta$.

Note that u and v are continuous for $z \neq 0$.

Furthermore, $\begin{cases} ru_r = -4r^{-4}\cos 4\theta = v_\theta \\ u_\theta = -4r^{-4}\sin 4\theta = -rv_r \end{cases}$

Hence $f(z)$ is differentiable for $z \neq 0$ with

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$
$$= e^{-i\theta}(-4r^{-5}\cos 4\theta + 4ir^{-5}\sin 4\theta)$$
$$= e^{-i\theta}(-4r^{-5}e^{-4i\theta})$$
$$= -4r^{-5}e^{-5i\theta}$$
$$= -4z^{-5}$$

c) $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$ $(r > 0, 0 < \theta < 2\pi)$

So $u(r,\theta) = e^{-\theta}\cos(\ln r)$, $v(r,\theta) = e^{-\theta}\sin(\ln r)$

Note that u & v are continuous on its domain.

Furthermore, $\begin{cases} ru_r = -e^{-\theta}\sin(\ln r) = v_\theta \\ u_\theta = -e^{-\theta}\cos(\ln r) = -rv_r \end{cases}$

Hence $f(z)$ is differentiable on its domain with

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$
$$= \frac{e^{-i\theta}(-e^{-\theta}\sin(\ln r) + ie^{-\theta}\cos(\ln r))}{r}$$
$$= \frac{i(e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r))}{re^{i\theta}}$$
$$= \frac{if(z)}{z}$$

8) From Ex. 7, we have

$$\begin{cases} u_x = u_r\cos\theta - u_\theta\frac{\sin\theta}{r} \\ v_x = v_r\cos\theta - v_\theta\frac{\sin\theta}{r} \end{cases}.$$

By the polar form of CR-equation, we have

$$u_x = u_r\cos\theta - u_\theta\frac{\sin\theta}{r} = u_r\cos\theta + v_r\sin\theta$$
$$v_x = v_r\cos\theta - v_\theta\frac{\sin\theta}{r} = v_r\cos\theta - u_r\sin\theta$$

Hence $f'(z_0) = u_x + iv_x$

$$= u_r\cos\theta + v_r\sin\theta + iv_r\cos\theta - iu_r\sin\theta$$

$$= \cos\theta(u_r + iv_r) - i\sin\theta(u_r + iv_r)$$
$$= (\cos\theta - i\sin\theta)(u_r + iv_r)$$
$$= e^{-i\theta}(u_r + iv_r)$$

9) a) Since $u_r = v_\theta / r$ and $v_r = -u_\theta / r$,

$$f'(z_0) = e^{-i\theta}(u_r + iv_r) = \frac{v_\theta - iu_\theta}{re^{i\theta}} = \frac{-i(u_\theta + iv_\theta)}{z_0}$$

b) $f(z) = z^{-1} = r^{-1}e^{-i\theta} = r^{-1}\cos\theta - ir^{-1}\sin\theta$

Hence $f'(z) = e^{-i\theta}(-r^{-2}\cos\theta + ir^{-2}\sin\theta)$

$$= -e^{-i\theta}(r^{-2})(\cos\theta - i\sin\theta)$$
$$= -r^{-2}e^{-2i\theta}$$
$$= -z^{-2}$$